LIFT IN A SHEARED FLOW

P.B.S. Lissaman*
AeroVironment Inc.
Monrovia, California

Abstract

The linearized field equations for a sail or wing in an inviscid flow with spanwise shear are developed and solutions derived for a flow varying linearly from the reflection plane to the masthead. An explicit lifting line solution for the unheeled case is given with lift distributions for typical sail geometry. The optimum loading to minimize induced drag for a given side force is derived, analogous to the well known elliptical loading of uniform flow. A powerful similarity solution is developed, which is capable of handling the general case of a heeled overlapping multiple sail plan. It is shown that the sheared flow loadings are significantly different from those given by the conventional quasi-uniform flow approximation.

Acknowledgements

The author takes great pleasure in noting the major contributions of three valued friends. Dr. John Letcher shared with him a decade-long struggle over this problem, and provided the deep theoretical knowledge and uncompromising analytical rigor which kept the author honest, at least to a reasonable approximation. Also gratefully acknowledged are the patience, acumen and good nature of Bart Hibbs and Ted Bate who co-endured many long evenings of painstaking detail in developing the computer programs required for the solutions presented.

Introduction

The sail of a yacht operates in a relative wind composed of the vector sum of the hull and the wind velocities. The latter varies strongly in magnitude with height, but can be considered approximately constant in direction. For a 12-m sailboat, typical conditions may be taken as a wind speed of 5.7 m/s 1.5 m from the surface increasing to 8.5 m/s 25 m from the surface, at masthead. Assuming a boat speed of 3.6 m/s at 30° to the true wind, this gives a relative speed at a height of 1.5 m of 9.0 m/s at 18° relative to hull direction of motion ignoring leeway and at 25 m of 11.7 m/s and 21.2°. Thus the relative masthead speed is approximately 1.3 times the speed near the foot of the sail (defined here as a shear of 0.30) with a twist of about 3.2°. This is typical of the sheared, twisted flow that the sail experiences. The wind twist is relatively small and, if accounted for in the kinematic boundary condition as a sail twist, may probably be neglected in the flow dynamics. The shear can not be ignored. It represents a significant magnitude of onset flow vorticity.

Because of the vortical nature of the flow the normal potential flow analysis of standard wing theory is no longer valid. While the standard singularities (sources, dipoles, vortex lines) exist in sheared flows, the classical expressions for their influence functions no longer apply since the Biot Savart Law is not valid, nor is the global form of the Bernoulli theorem. Thus existing wing theory models are incorrect, and the magnitude of the inaccuracy is unknown.

To date, most approaches to the sail in a sheared flow have used the uniform flow or the quasi-uniform flow approximation. In the uniform flow model a representative mean flow is assumed which is constant with height and the problem is treated as a potential flow. In the quasi-uniform flow model a general singularity system (typically a vortex lattice) is constructed and assumed to be immersed in a nonuniform but homenergetic (irrotational) onset flow. Consequently, the classical potential flow influence functions of the singularities may be used and a solution may be determined.

The uniform flow model has the advantage of being theoretically an exact solution to the assumed model -- however the model is evidently an approximation to the actual flow kinematics and probably significantly in error in regions where the onset flow is different from the assumed flow. The quasi-uniform flow model is probably in some sense a first order approximation for small shear values but is unsatisfactory because the flow kinematical model is based on theoretical inconsistencies so that again errors of unknown magnitude are introduced.

While it is possible to construct a detailed multiple element heeled lifting surface analysis for either of the above models, it appears that the precision of such models is unjustified because of the intrinsic incorrectness of the fundamental elements of the model. Either the onset flow or the singularity induced flow is incorrectly treated. In either case, the solution is not accurate enough to provide any useful design tool. It has never been proved that an elaborate lifting surface model justifies the complexity and expense.

Theoretical Background

The theoretical background to this analysis is based on Von Karman and Tsien[1] and Lighthill[2]. Both papers develop the fundamental linearized equations used as the starting point of the present paper. Von Karman and Tsien then approach the specific problem of the wing in a sheared flow while Lighthill employs the more basic approach of seeking to analytically express the fundamental singularities in a sheared flow. Neither paper gives an explicit solution method for the present problem but both provide very valuable insights

*Vice President, Aerosciences

into the fundamental difficulties -- essentially that the analytical expressions for the perturbations do not appear to converge at the downstream infinity. Von Karman and Tsien (in Section 3 of their paper) state that they believe that this does not compromise the accuracy of the theory and is simply a consequence of the linearization. Lighthill discusses the issue at some length and proposes different expansions near and far from the disturbing body.

If we consider the specific problem of the present paper and attempt to qualitatively describe the fluid mechanics, it appears that a steady-state downstream surface of discontinuity, (the Trefftz Plane) which is normally assumed in steady wing theory when vortex sheet stability is ignored cannot exist in the present case. This is because the nonuniform vortical onset flow will cause continuous distortion of the wing vortex wake as it proceeds downstream, as well as perturbing the initial free-stream vorticity distribution. We have no means of determining this effect in the present analysis. It would be of great interest (and considerable effort) to set up an unsteady finite difference theoretical analysis in which the wake and the free-stream vorticity is permitted to convect as the flow developes. It would also be of considerable interest to conduct a well-controlled experiment of a wing in a sheared flow to determine the importance and magnitude of this effect in a real flow. For the present paper we will use the equations of Von Karman and Tsien and develop an operational solution from that basis.

Analysis

General Field Equation

We now set up the field equations for a lifting surface in a sheared flow. Because most of the literature relates to the lifting aircraft wing we will use standard notation for that case with the equivalent sail parameter in parenthesis. Figure 1 shows the geometry. The x axis is parallel to the main flow (relative wind) $U(y,z)$ with the y axis normal to the plane of symmetry (water plane) and in the direction of the right wing tip (mast head) for a wing with no dihedral (heel). The z axis is positive in the direction of the aerodynamic force (to leeward). The perturbation velocities are u, v, w, the pressure, p, and density ρ.

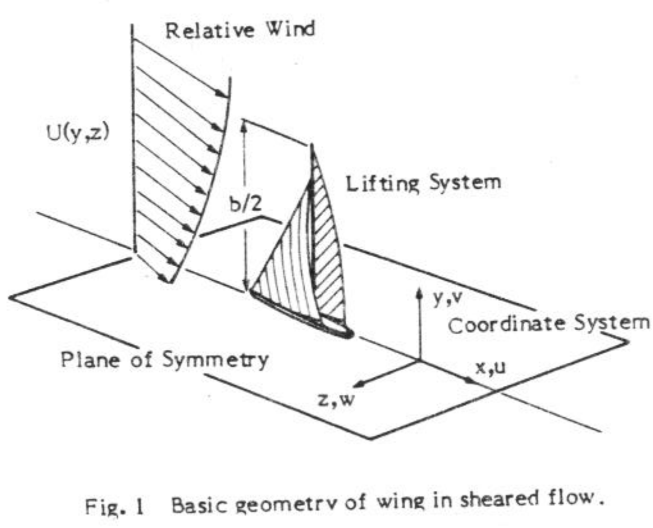

Fig. 1 Basic geometry of wing in sheared flow.

Now for a steady, incompressible inviscid flow the linearized Euler equations become

$$Uu_x + vU_y + wU_z = -\frac{1}{\rho} p_x \quad (1)$$

$$Uv_x = -\frac{1}{\rho} p_y \quad (2)$$

$$Uw_x = -\frac{1}{\rho} p_z \quad (3)$$

Here the suffix indicates partial differentiation. The continuity equation becomes

$$u_x + v_y + w_z = 0 \quad (4)$$

The velocity perturbations may be eliminated to give

$$(p_x/U^2)_x + (p_y/U^2)_y + (p_z/U^2)_z = 0 \quad (5)$$

while equations (2) and (3) can be combined to form a vorticity-like equation

$$(Uv)_z - (Uw)_y = 0 \quad (6)$$

Thus far we have followed the development of von Karman and Tsien who then proceed to propose a formal solution of Equation (5) for the lifting line case but do not provide an explicit operational expression for this solution.

For our analysis, to give explicit solutions, we assume $U = U(y)$ and then substitute Equation (2) into (5) to give

$$v_{xx} + v_{yy} + v_{zz} = (U_{yy}/U)v \quad (7)$$

We now note that putting $U_{yy} = 0$ converts equation (7) to the LaPlace equation

$$v_{xx} + v_{yy} + v_{zz} = 0 \quad (8)$$

The mainstream velocity is now given by

$$U = U_o(1 + 2K\, y/b),\ y > 0 \quad (9)$$

where $b/2$ represents the semispan (mast height) and K the dimensionless shear factor. For convenience we will define the speed factor V by $V = U/U_o$. We note that conditions of symmetry give $V = 1$ at $y = 0$ implying that the image freestream is given by $V = 1 - 2Ky/b$, $y < 0$.

It is now possible to develop exact solutions to the above equations, both for the lifting line and lifting surface case, including the effects of dihedral (heel).

Lifting Line Solutions

We will consider the zero dihedral case. In the Trefftz plane, we obtain the two-dimensional LaPlace equation

$$v_{oyy} + v_{ozz} = 0 \quad (10)$$

where the suffix o indicates far downstream conditions

At the wing, as shown by von Karman and Tsien we obtain the local lift, l, as

$$l = U^2 c\, C_l/2 \quad (11)$$

where $\rho = 1.0$ is used in all following equations where c is the local chord and C_l the local lift coefficient. The spanwise gradient of lift is given by

$$l_y = 2Uv \tag{12}$$

The downwash in the Trefftz plane is given by $-w_o$ so that the induced angle at the wing becomes $-(w_o/U)/2$.

Then the standard equation of lifting line theory becomes

$$\alpha(y) = l(y)/(\pi u^2 c) - (wo/U)/2 \tag{13}$$

where α is the airfoil effective angle of attack at each station taking into account the angle of incidence, the angle of twist and the angle of effective camber (by using the zero lift line as a reference).

We now generate a harmonic function $\tilde{v}_o - i\tilde{w}_o$ and note that putting $\tilde{v}_o \equiv v_o$ will satisfy the field equation and thus provide the sidewash solution for the sheared flow.

Expression of Harmonic Functions

Equation (10) can be satisfied by any harmonic function satisfying the boundary conditions.

It is convenient to use the Glauert[3] method where a suitable choice for the complex potential function $F(\zeta) = \tilde{\varphi} + i\tilde{\psi}$ where $\zeta + 1/\zeta = 4t/b$, $t = y + iz$ is $F(\zeta) = (iU_o b/2)\sum B_n \zeta^{-n}$ with n odd.

It is now assumed that $dF/dt = \tilde{v} - i\tilde{w}$ so that on (or just above) the axis, $z = o^+$, $o < y < b/2$ the following expressions apply:

$$\tilde{\varphi} = (U_o b/2)\sum B_n \sin n\theta = (U_o b/2)S_1 \tag{14}$$

$$\tilde{\psi} = (U_o b/2)\sum B_n \cos n\theta = (U_o b/2)S_2 \tag{15}$$

where $\tilde{\varphi}(\theta = 0) = 0, \tilde{\psi}(\theta = \pi/2) = 0$.

$$\tilde{v} = -U_o \frac{\sum nB_n \cos n\theta}{\sin\theta} = -U_o S_3$$

$$= \tilde{\varphi}_y = \tilde{\psi}_z \tag{16}$$

$$\tilde{w} = -Uo \frac{\sum n B_n \sin n}{\sin\theta} = -U_o S_4 \tag{17}$$

$$= \tilde{\varphi}_z = -\tilde{\psi}_y$$

where $\cos\theta = 2y/b$

An additional required function is $\tilde{\Phi}$, the integral of $\tilde{\varphi}$, defined by $\tilde{\Phi}_y = \tilde{\varphi}$, $\tilde{\Phi}(\theta = 0) = 0$. This is given by

$$\tilde{\Phi} = \frac{-Uob^2}{4}\ \frac{1}{2}\left[B_1\left(\theta - \frac{\sin 2\theta}{2}\right) + \sum_3 B_n \left(\frac{\sin(n-1)\theta}{n-1} - \frac{\sin(n+1)\theta}{n+1}\right)\right]$$

$$= \frac{-U_o b^2}{4} S_5 \tag{18}$$

Development of Aerodynamic Functions

We now note that putting $v \equiv \tilde{v}_o$ provides the solution to equation (10) the field equation defining the sidewash. It is now necessary to derive the other aerodynamic functions

The downwash, $-w$, is obtained from equation (6)

$$(Uw)_y = (Uv)_z = U\tilde{\varphi}_{yz}$$

$$Uw = U\int \tilde{w}_y\, dy = U\tilde{w} - U_y \int \tilde{w}\, dy$$

$$= U\tilde{w} + U_y \tilde{\psi}$$

$$w = \tilde{w} + (U_y/U)\tilde{\psi} \tag{19}$$

The lift, l, is obtained from equation (12)

$$l_y = 2U\tilde{\varphi}_y$$

$$= 2U\tilde{\varphi} - 2U_y \int \tilde{\varphi}\, dy$$

$$= 2U\tilde{\varphi} - 2U_y \tilde{\Phi} \tag{20}$$

This yields $C_l = \frac{4\tilde{\varphi}}{Uc} - \frac{4U_y}{U^2}\ \frac{\tilde{\Phi}}{c}$

The Lifting Line Equation

We now substitute into equation (13) to give

$$\alpha = \frac{2\tilde{\varphi}}{\pi Uc} - \frac{2U_y}{\pi U^2 c}\tilde{\Phi} - \tilde{w}/2U - U_y\tilde{\psi}/2U^2$$

$$\alpha = \frac{A}{\pi}\ \frac{S_1}{VC} + K\frac{A}{\pi}\ \frac{S_5}{V^2 C} + \frac{S_4}{2V} - \frac{K}{2}\ \frac{S_2}{V^2}$$

where C is the normalized chord defined by $c/\bar{c}$ where $\bar{c}$ is the mean chord. The aspect ratio, A, is defined by $b/\bar{c}$.

The simplest form of the equation now is

$$V\alpha = \frac{A}{\pi}\ \frac{S_1}{C} + \frac{S_4}{2} + \frac{K}{V}\left[\frac{A}{\pi}\ \frac{S_5}{C} - \frac{S_2}{2}\right] \tag{21}$$

where S_1, S_2, S_4, S_5 represent series containing B_n and various trigonometric functions of θ. The normalized chord, C, and angle of attack, α, are arbitrary functions of spanwise position, θ. The normalized speed V is defined by $V = 1 + K\cos\theta$.

The equation (21) can be solved by any standard method to obtain the coefficients B_n.

The spanwise lift distribution is now given by

$$l = U_o^2 b\,(VS_1 + KS_5) \tag{22}$$

The spanwise induced drag distribution, d, is given by

$$d = \frac{U_o^2 b}{2}\left(S_1 + \frac{K}{V}S_5\right)\left(S_4 - \frac{K}{V}S_2\right) \tag{23}$$

Comparison of Different Approximations

It is now possible to identify the lifting line equations employed by the various models, since the uniform flow and quasi-uniform flow models are seen to be approximations of the sheared flow model (equations 21, 22). The former are shown below.

Uniform Flow Model

$$\bar{V}\alpha = \frac{A}{\pi}\,\frac{S_1}{C} + \frac{S_4}{2}\;;\; l = U_o^{\,2} b\,\bar{V}\,S_1 \tag{24}$$

where $\bar{V}$ is some mean speed ratio chosen to be most representative

Quasi-Uniform Flow Model

$$V\alpha = \frac{A}{\pi}\,\frac{S_1}{C} + \frac{S_4}{2}\;;\; l = U_o^{\,2} b\,V\,S_1 \tag{25}$$

It is of interest to note that the quasi-uniform flow model can be solved by applying the uniform flow lifting line equation (24) to a similar case with an effective twist of $V\alpha$.

Optimal Loading Case

As shown by Karman and Tsien the optimal loading occurs for spanwise constant downwash angle, the same condition as with uniform flow. Assume this angle is γ. Then equation (6) yields

$$-(U^2\gamma)_y = (Uv)_z$$

$$-2\,U_y\,U\gamma = U\,v_z = U\,\tilde{v}_z = U\,\tilde{w}_y$$

$$\tilde{w}/\gamma = \frac{-4\,KyU_o}{b} + U_o$$

$$= -U_o\left[1 + 4\,Ky/b\right],\; y > o \tag{26}$$

Thus the sidewash function, $\tilde{v}$, must be the harmonic conjugate of the above ramp-type function. Another interpretation of $\tilde{v}$ is that, in wing theory, it is the shed vorticity associated with the symmetrical double ramp-type downwash. This gives rise to a characteristic saddleback type of spanwise load distribution which is derived below.

Writing the downwash $\tilde{w}$ in terms of the circular coordinate θ we obtain

$$\tilde{w} = -\gamma U_o\,(1 + 2\,k\cos\theta)$$

It can be shown that the harmonic conjugate $\tilde{v}$ is given by

$$\tilde{v} = -\gamma U_o\left[\cot\theta + 2K(\cos\theta\,L^*/\pi + 2\cot\theta/\pi)\right]$$

where

$$L^* = \ln\left[(1 - \sin\theta)/(1 + \sin\theta)\right] \tag{27}$$

The other functions $\tilde{\varphi}$ and $\tilde{\Phi}$ analogous to equations (14, 18) are given by

$$\tilde{\varphi} = \frac{\gamma U_o b}{2}\left[\sin\theta - \frac{2K}{\pi}\left(\frac{\cos^2\theta}{2}L^* - \sin\theta\right)\right] \tag{28}$$

$$\tilde{\Phi} = -\gamma\frac{U_o b^2}{4}\left[\frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + \frac{2K}{\pi}\left\{\frac{\cos^3\theta}{6} \times L^* + \frac{\sin 2\theta}{12} + \theta/6 + \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right\}\right] \tag{29}$$

The spanwise lift distribution is now obtained by substituting into equation (20) providing

$$l/(\gamma U_o^{\,2} b) = V\left[\sin\theta - \frac{2K}{\pi}\left(\frac{\cos^2\theta}{2}L^* - \sin\theta\right)\right] + K\left[\frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + \frac{2K}{\pi}\left\{\frac{\cos^3\theta}{6}L^* + \frac{\sin 2\theta}{12} + \theta/6 + \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right\}\right] \tag{30}$$

The optimal spanwise loading is shown in Figure 2, where the loading has been normalized by the mean load. The local loading can be integrated to provide the total lift, L, defined as the load on the lifting surface on one side of the plane of symmetry.

$$L/(\gamma U_o^{\,2} b^2) = \frac{1}{2}\left(\pi/4 + \frac{4}{3}K + \frac{2}{\pi}K^2\right) \tag{31}$$

It is of interest to show the local lift coefficients required to achieve optimal loading. Using a representative triangular sail plan form, we show the normalized local lift coefficient for optimal loading in Figure 3.

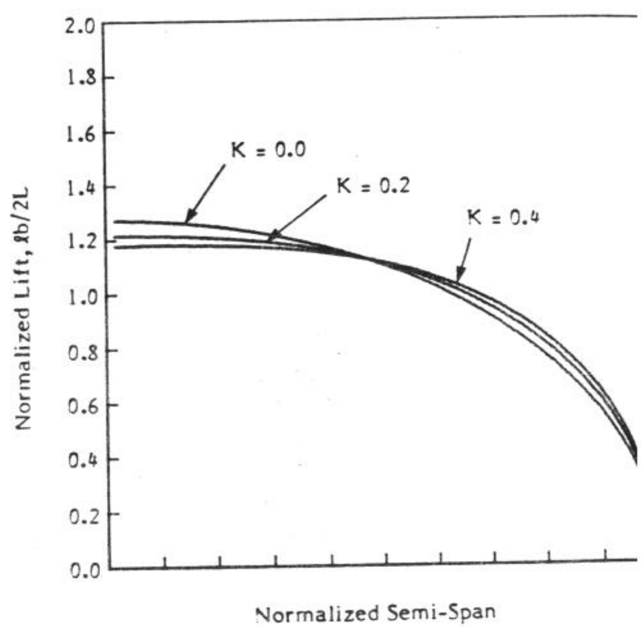

Fig. 2 Optimal Loading for Different Shears.

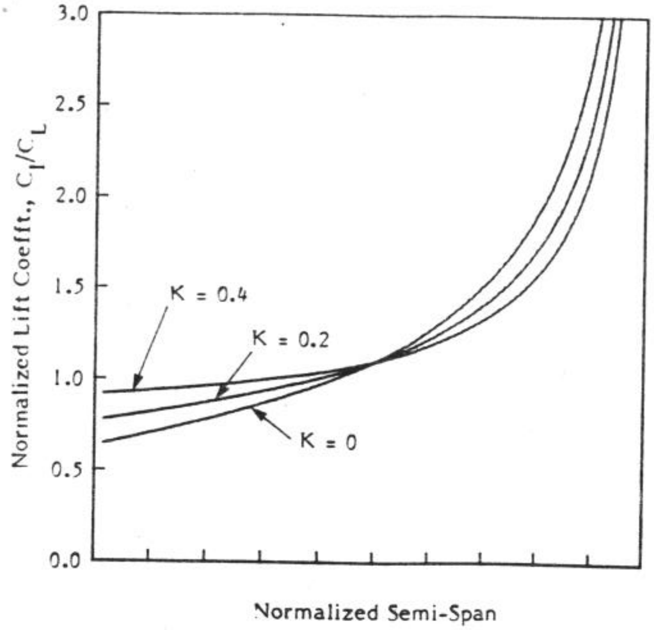

Fig 3. Optimal Local Lift Coefficient on Triangular Sail.

The induced drag is given by $D_i = \gamma L/2$. We calculate the induced drag efficiency e in the same way as for a monoplane wing. Thus

$$e = L^2/(\pi D_i \, qb^2)$$

where q is some representative dynamic pressure. If, as for C_l we take the representative case as at half the semispan (half mast height), $y = b/4$ we obtain

$$e = \left(1 + \frac{16K}{3\pi} + \frac{8}{\pi^2} K^2\right)/(1 + K/2)^2 \qquad (32)$$

This indicates that the span efficiency for an optimally loaded wing in a sheared flow when based on average flow speed is higher by approximately the shear factor from that for the uniform flow case. This is illustrated in the Table 1.

TABLE 1. Optimal span efficiency for sheared flows.

Shear Factor K	Span Efficiency
0	1.0
0.1	1.068
0.2	1.134
0.3	1.196
0.4	1.256

Sheared Flow Similarity Law

We now show that any given sheared flow can be converted to a similar potential flow in a stretched coordinate system and solved by standard methods. The similar solution may then be converted to the corresponding sheared flow.

The similarity approach appears to have considerable versatility. In order to illustrate the method we describe first how a sheared flow solution can be derived from a given potential flow solution. Consider a lifting surface of general plan form and zero thickness in a uniform flow and assume that the perturbation potential $\tilde{\varphi}(x,y,z)$ has been determined with velocity components $\tilde{u},\tilde{v},\tilde{w}$, satisfying the usual kinematic boundary conditions on the wing and the dynamic vortex sheet boundary condition on the wake. Evidently the sidewash $\tilde{v}(x,y,z)$ satisfies the LaPlacian $\nabla^2\tilde{v} = o$. Now assume this same sidewash component occurs for a different wing in a sheared flow, $U = U(K,y)$ where it satisfies the field equation. We will define the components of the sheared flow to be u,v,w and note that by the similarity $\tilde{v}(x,y,z) \equiv v(x,y,z)$. Thus v is the sidewash function defined in all space for a certain wing in a sheared flow and we can now determine other flow field components of this new wing. In principle, this can be accomplished once $v(x,y,z)$ is known by satisfying the pseudo-vorticity equation (6) $(Uw)_y - (Uv)_z = 0$ and the continuity equation (8), $u_x + v_y + w_z = 0$. It is noted that this is a simple process of quadrature in which the pseudo-vorticity equation provides a pointwise algebraic (not differential) equation for the velocity derivative w_y and hence w, while the continuity equation then provides a similar equation for the velocity derivative u_x. It is noted that the sheared flow wing has the same planform but a different twist and loading where the perturbation $w(x,y,z)$ defines the angle of attack and camber and $u(x,y,z)$ and $v(x,y,z)$ define the loading.

We now show for a given wing in a sheared flow how to construct the similar geometry of the potential flow wing and then from the potential solution how to derive the loading in the sheared flow case.

We use normal symbols to represent the sheared flow quantities and symbols with a tilda overbar for the similar quantities. In the similar case, typical boundary conditions will be $\tilde{w}(x,y) = U_o \, \tilde{s}(x,y)$ defining the downwash $\tilde{w}$ and slope $\tilde{s}$ of the lifting system in a uniform flow U_o with the usual vortex sheet conditions in the wake. Assume that the solution for these boundary conditions is known then the potential $\tilde{\varphi}(x,y,z)$ is also known as well as the streamwise perturbation flow, the spanwise flow and the downwash, $\tilde{u},\tilde{v},\tilde{w}$.

Now there exists a similar sheared solution in a nonuniform flow $U = U(y)$ having the same spanwise flow so that $\tilde{v}(x,y,z) = v(x,y,z)$. We now develop the other flow quantities in the sheared flow. Equation (6) now yields

$$(Uw)_y = (Uv)_z$$
$$= Uv_z = U\tilde{v}_z = U\tilde{w}_y.$$

This yields

$$(Uw)_y = U\tilde{w}_y.$$

which provides

$$w = \tilde{w} - (U_y/U)\int \tilde{w}\, dy$$
$$\tilde{w} = w + \int (U_y/U)\, w \, dy.$$

The slope relationship is given by

$$\tilde{s} = Vs + U_y/Uo \int_0^y s\,dy \qquad (33)$$

$$Vs = \tilde{s} - U_y/Uo\ V \int_0^y \tilde{s}\,dy$$

The chordwise scaling remains the same, thus in the potential flow problem all vertical, z, distances must be scaled as indicated by equation (33).

To determine the pressure differential on a lifting element, we note that for a general dihedral (heeled) surface of local dihedral angle $\delta(s)$ as shown in Figure 4, there is a discontinuity in both v, w and V_t, the tangential velocity, across the surface but (for zero thickness surface) none in V_n, the normal velocity. The spanwise variation in δ may be taken to represent a 'bendy' mast although including this refinement is probably not consistent with other approximations. The discontinuities are expressed as follows

$$\Delta V_s = \Delta v \cos\delta + \Delta w \sin\delta$$

$$0 = -\Delta v \sin\delta + \Delta w \cos\delta$$

where Δv, Δw represent the difference between the upper and lower surface values of v, w. We note that the spanwise gradient in pressure difference Δp is given by

$$-\frac{\partial}{\partial s}(\Delta p) = U \frac{\partial}{\partial x} \Delta V_s$$

$$-\frac{\partial}{\partial s}(\Delta p) = 2U \frac{\partial}{\partial x}(\Delta v)\ /\cos\delta$$

$$-\frac{\partial}{\partial s}(\Delta p) = 2U \frac{\partial}{\partial x}(v)\ /\cos\delta$$

$$-\frac{\partial}{\partial s}(\Delta p) = 2U\tilde{\varphi}_{yx}/\cos\delta$$

$$= 2U \frac{\partial}{\partial s} \tilde{\varphi}\,x/\cos^2\delta$$

We integrate this noting that Δp at the tip (s_t) must be zero and obtain

$$-\Delta p/2 = U\tilde{\varphi}_x/\cos^2\delta - \int_{st}^{s} U_y \tilde{\varphi}_x\ ds/\cos\delta$$

$$\Delta p = V\Delta\tilde{p} - U_y/U_o \int_{st}^{s} \Delta\tilde{p}\cos\delta\,ds \qquad (34)$$

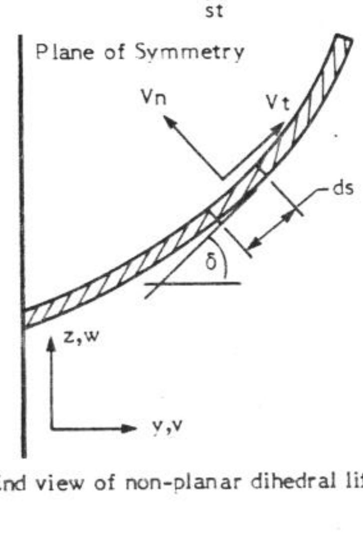

Fig. 4 End view of non-planar dihedral lifting surface.

The similarity procedure is thus as follows: Construct a similar lifting surface geometry of the same planform and heel where all z airfoil distances (parallel to the reflection plane) such as camber, twist and angle of attack have been increased by the integral relationship of equation (33). Then solve the potential problem for a uniform onset flow of U_o, to arrive at the similar potential solution. Finally, modify all the potential pressures $\Delta\tilde{p}$ by the integral relationship expressed by equation (34) to obtain the actual pressures.

The similarity approach can be used to develop a lifting line model similar to that given previously.

In principle this should be accomplished as follows: the actual angle of attack of the wing should be modified according to equation (33) providing a similar potential flow wing of different twist. This is then solved by uniform flow lifting line theory as expressed by equation (24). The resulting load distribution is then modified according to equation (34).

For the case of a constant chord wing of zero dihedral manipulation of the equations recovers the sheared flow solution given by equation (21). For varying chord we have not been able to recover equation (21). This is evidently due to subtleties in the boundary condition of equation (34) where the spanwise integration must be adjusted to give zero load at the trailing edge, which in general is not at the wing tip. It is believed that a more careful attention to the boundary condition and order of spanwise and chordwise integration of the pressure difference Δp will resolve this. It will be most convenient to investigate this directly using a numerical lifting surface solution. Work is in progress on this issue.

Comparison of Sheared Model With Previous Models

It is of interest to compare the sheared lifting line model developed here with the two conventional models. This will give an indication of the accuracy of the latter models and hence a reasonable engineering criterion of how much geometrical structure is worth including in the approximate models and also, importantly, what the magnitude and source of the error of the popular quasi-uniform flow model is.

For this purpose it is convenient to use the simplest representative model and geometry. Thus we will choose the zero dihedral (unheeled) lifting line model and the geometry of a triangular planform of aspect ratio 7.0. This geometry can be seen in Figure 5, where it is overlaid on a typical modern 12-m sail plan.

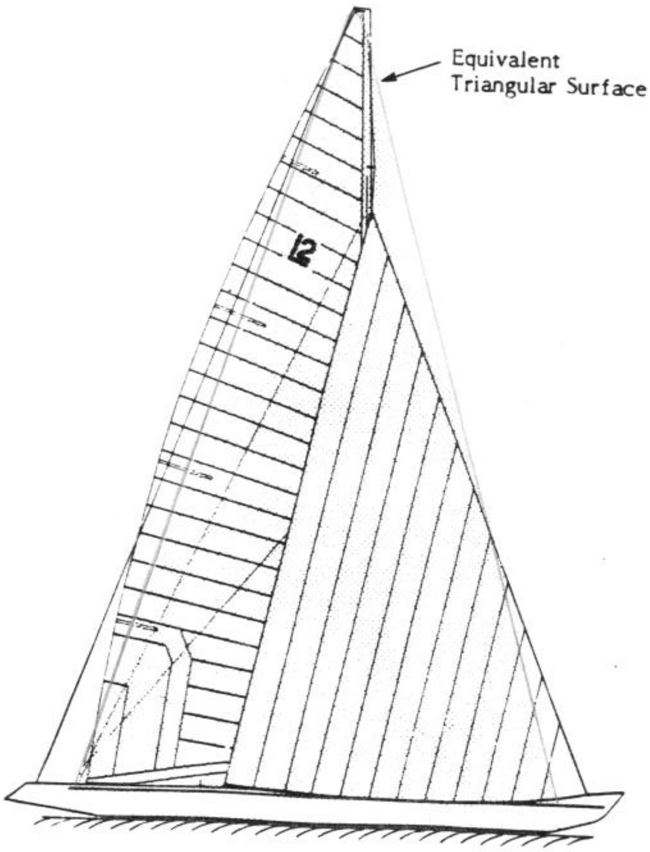

Fig. 5 Typical 12-M sail plan compared with triangular surface.

The lift boundary conditions at the tip (masthead) and root (sail foot) require careful discussion. Most real sail heads are pointed and thus, if untwisted, would imply infinitely large local lift coefficients according to lifting line theory. Generally, the membrane compliance near the head is such that in any real sail the lifting surface is highly twisted to reduce effective angle of attack thus it is washed out (slackened) so that the tip loading is reduced. However, the pointed tip is a geometrical reality of most sails and must be part of any model.

At the foot an interesting situation occurs regarding the aerodynamic continuation of the sails to the waterplane. Sea test photography indicates that when close-hauled the foot of the fore sail of many types will usually overlap and closely match the hull, so that it appears to be an adequate assumption to assume that the fore sail is continued aerodynamically through the canoe body to the waterplane. The main sail experiences a different situation. On some racing classes great effort is made to droop the boom so that the clew is virtually at deck height and the main sail foot may be approximately sealed. On other classes, while the fore sail may be sealed, there is a very distinct gap between the foot of the main and the deck. This case should presumably be treated as a chord discontinuity. This can readily be done using the discontinuous functions lifting line shown by Lissaman[(4)]. This is likely to have a significant effect on induced drag but will not be dealt with here.

For the model under discussion here, we will assume that both the fore and main sail are effectively aerodynamically sealed, since our object is to study discrepancies between models using the most realistic simple geometry.

A comparison of the three cases is shown in Figure 6 where the unheeled, untwisted, $A = 7.0$, $K = 0.3$ case has been shown. The differences in spanwise loading are seen to be quite significant with the sheared flow loading falling approximately between the quasi-uniform and uniform case. The latter overpredicts at the foot and underpredicts at mast head as might be expected. The quasi-uniform model underpredicts at all stations, being about 10% low at the foot and 5% low overall. The sheared flow lift slope based on dynamic pressure at mid mast height is 4.58 which may be compared to that of an elliptical wing of the same aspect ratio in a uniform flow which is 4.89. The differences in induced drag may also be determined, but are not reported here since these calculations are not complete.

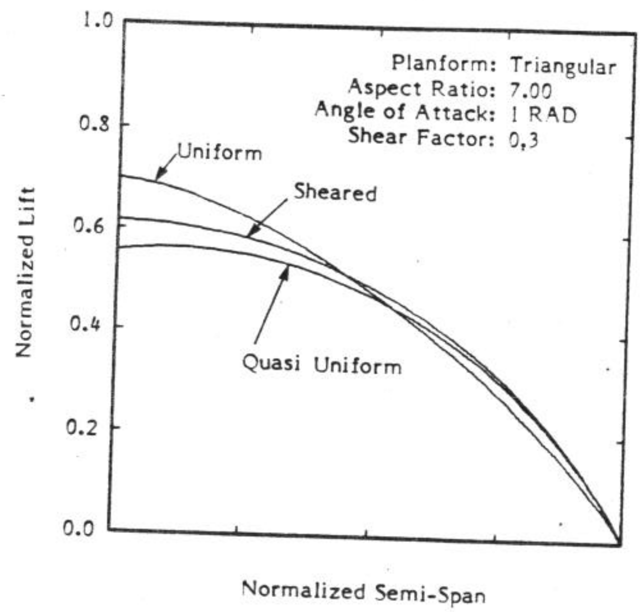

Fig. 6 Spanwise loading predicted by different models.

Extension to General Case

It is noted that the most appropriate general case should include twisted multiple lifting surfaces with overlap and pointed tips, dihedral (heel) and some model for reflection and lift continuation through the canoe body to the image as well as a model for chord discontinuity associated with the gap between the main boom and fore sail foot and the deck.

The sheared lifting line solution expressed in equations (21, 22) very efficiently provides results yielding an important insight to many of the above features, particularly the effect of planform, twist and gaps at the foot of the sail. The effect on induced drag of the twist and gap is of particular interest. Calculations for these cases are in progress. It is noted, however, that the sheared lifting line solution cannot deal explicitly with the effect of heel or of the multiple overlapping lifting surfaces.

The sheared similarity solution expressed in equations (33, 34) is capable, in principle, of dealing with all the geometrical effects described in the first paragraph. Evidently, it requires a potential flow multiple lifting surface program to provide the similarity solution to the stretched geometry. These programs are available, and currently work is in progress to develop numerical results.

Conclusions

A method has been developed to solve the problem of the lifting wing in a linearly sheared flow and to determine the optimum loading to minimize induced drag. This solution is intrinsically more accurate than current quasi-uniform flow methods since it properly includes the vortical nature of the onset flow. A relatively simple solution has been given for the unheeled lifting line. It is noted that solutions for geometry representative of a 12-m sail plan are significantly different from those obtained using a uniform flow or a quasi-uniform flow approximation.

A new similarity solution has been developed which in principle is capable of accurately solving the general heeled multiple gapped overlapping surface sail plan providing that this geometry can be solved in a potential flow. Calculations for this case are not yet complete.

It is noted that there are still theoretical inconsistencies in the present solutions as well as in existing quasi-uniform approaches which require further analysis. Indeed, there may be no Trefftz Plane steady state solution for a wing in a vortical flow. However, in spite of these inconsistencies the methods presented here will be more accurate than existing potential flow type solutions.

Recommendations

It is recommended that further work in investigati the solution techniques developed here be done, particularly in resolving apparent discrepancies between the present lifting line method and the present lifting surface similarity approach.

The present solution method, once validated, should be exercised to determine the effects of discontinuities at sail foot and of the pointed sail head. The method can then be implemented to develop desirable loading distributions for design of optimal real sails.

It is further recommended that an unsteady sheared flow analysis be developed to determine theoretically the effect of vorticity convection which has been ignored here. In addition to the theoretical analysis, it will be very important to conduct well-controlled tests of surfaces of simple geometry to experimentally determine the practical magnitude of the nonlinear inviscid convection effects.

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