Effect of Flow Shear on Induced Drag

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Von Kármán and Tsien[1] formulated the lifting line theory for a wing in nonuniform flow in 1945. In their formulation, the approaching flow may have arbitrary variations in the plane perpendicular to the flight direction. By working with the perturbation pressure rather than the perturbation velocities, they have obtained the condition for the minimum induced drag and the method of calculation for the induced drag of a lifting line with arbitrary spanwise lift distribution. No specific examples, however, were given. In this note, we present the calculation of the induced drag of an elliptically loaded lifting line in an exponentially sheared flow in the vertical direction. The result applies equally well to a linearly sheared flow provided the shear rate is small.

Consider a lifting line with a unit semispan in an incompressible, inviscid stream which is sheared in the vertical direction. Let the rectangular coordinate system be such that the y axis is in the direction of the span with the lifting line extending from $y = -1$ to $y = +1$, the x axis in the direction of the approaching flow $U(z)$, and the z axis pointing upwards. By considering the flow in the Trefftz plane ($x \to \infty$), von Kármán and Tsien[1] give the following governing equation:

$$(\partial/\partial y)[(1/U^2)(\partial\phi/\partial y)] + (\partial/\partial z)[(1/U^2)(\partial\phi/\partial z)] = 0 \quad (1)$$

in which the "potential function" ϕ is related to the perturbation velocities v and w in the y and z direction as follows:

$$v_1 = v(x \to \infty, y, z) = (1/\rho U)(\partial\phi/\partial y) \quad (2)$$

$$w_1 = w(x \to \infty, y, z) = (1/\rho U)(\partial\phi/\partial z) \quad (3)$$

with density denoted by ρ. The boundary conditions on ϕ are

$$\phi = 0 \qquad \text{for } |y| \qquad \text{and } |z| \to \infty \quad (4)$$

$$\phi = \pm l(y)/2 \qquad \text{for } z = 0 \pm \quad (5)$$

in which $l(y)$ is the spanwise lift distribution. Since the downwash at the lifting line is one half of that at $x = \infty$[1], the induced drag is

$$D_i = -\frac{1}{2\rho}\int_{-1}^{1}\left(\frac{1}{U^2}\frac{\partial\phi}{\partial z}\right)_{z=0} l(y)dy \quad (6)$$

Once ϕ is known, the induced drag may be evaluated according to the foregoing expression.

For $U(z) = U_0 e^{Kz}$, Eq. (1) becomes

$$(\partial^2\phi/\partial y^2) + (\partial^2\phi/\partial z^2) - 2K(\partial\phi/\partial z) = 0 \quad (7)$$

We remark here that if $U(z) = U_0(1 + Kz)$, $K \ll 1$, we would obtain the same equation for ϕ if terms of $0(K^2)$ are neglected. Using the Fourier cosine transform, we obtain the solution for the upper half-space $z > 0$:

$$\phi = \int_0^\infty f(\lambda)\exp\{[K - (K^2 + \lambda^2)^{1/2}]z\}\cos\lambda y d\lambda \quad (8)$$

For elliptic loading

$$l(y) = (2L/\pi)(1 - y^2)^{1/2} \quad (9)$$

where L is the total lift. Apply the boundary condition (5)

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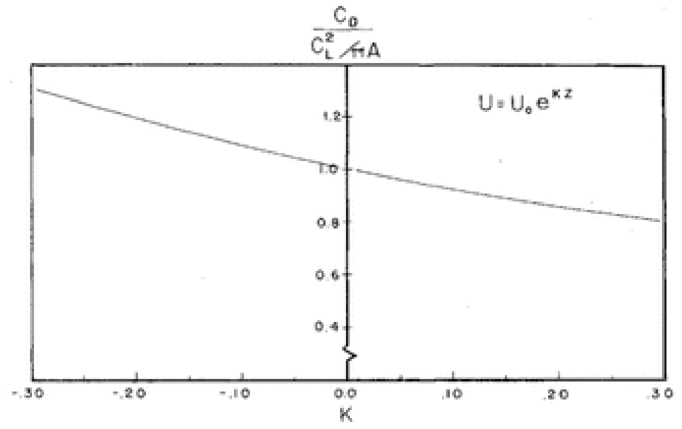

Fig. 1 The effect of exponential flow shear on the induced drag.

for $z = 0+$; we obtain $f(\lambda)$ by inversion in terms of Bessel function[2]:

$$f(\lambda) = \frac{2}{\pi}\int_0^\infty \frac{L}{\pi}(1 - y^2)^{1/2}\cos\lambda y dy = \frac{L}{\pi}\frac{J_1(\lambda)}{\lambda} \quad (10)$$

The induced drag becomes

$$D_i = -\frac{L^2}{\pi^2\rho U_0^2}\int_{-1}^{1}(1 - y^2)^{1/2}dy \times \int_0^\infty [K - (K^2 + \lambda^2)^{1/2}]J_1(\lambda)\cos\lambda y\frac{d\lambda}{\lambda} \quad (11)$$

The integral involving K alone can be evaluated directly, since[2]

$$\int_0^\infty J_1(\lambda)\cos\lambda y\frac{d\lambda}{\lambda} = (1 - y^2)^{1/2} \qquad \text{for } |y| < 1$$

The other integral may be simplified by a change in the order of integration, and by the use of the Poisson's integral representation of Bessel functions.[3] We finally obtain the induced drag coefficient normalized with respect to U_0^2:

$$C_{D_i} = \frac{C_L^2}{\pi A}\left\{-\frac{8K}{3\pi} + 2\int_0^\infty (K^2 + \lambda^2)^{1/2}J_1^2(\lambda)\frac{d\lambda}{\lambda^2}\right\} \quad (12)$$

in which the aspect ratio is denoted by A. For $K = 0$,

$$\int_0^\infty J_1^2(\lambda)\frac{d\lambda}{\lambda} = \int_0^\infty [J_0(\lambda) - J_1'(\lambda)]J_1(\lambda)d\lambda = \frac{1}{2}$$

the well-known result is recovered. For $|K| \to \infty$, the integral may be approximated by

$$2|K|\int_0^\infty \frac{J_1^2(\lambda)}{\lambda^2}d\lambda = \frac{8}{3\pi}|K|$$

We therefore obtain the result that

$$\frac{\pi A C_{D_i}}{C_L^2} \to \begin{cases}(16/3\pi)K & K \to -\infty \\ 0 & K \to \infty\end{cases}$$

The integral in Eq. (12) has been evaluated by the use of a digital computer. The results are shown in Fig. 1 for values of K varying from -0.3 to $+0.3$. It is seen that for the approach velocity increasing with height, the induced drag is diminished and vice versa.

References

[1] von Kármán, T. and Tsien, H. S., "Lifting-line theory for a wing in non-uniform flow," Quart. Appl. Math. **3**, 1–11 (1945).

[2] Oberhettinger, F., *Tabellen zur Fourier Transformation* (Springer-Verlag, Berlin, Germany, 1957), pp. 6 and 61.

[3] Erdélyi, A., *Higher Transcendental Functions* (McGraw-Hill Book Company Inc., New York, 1953), Vol. II, p. 14.